Numerical solution of Mild-slope equation using Virtual Element Method

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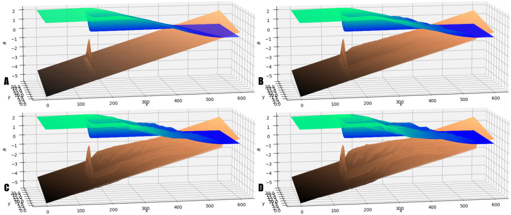
FOREWORD

- A parallel project to my PhD,
- Virtual element method of order k with Robin's Boundary condition,
- Application to a concrete problem.



My main PhD work

Photo of the port of Cherbourg (France)

Overview

1. Model problem
2. Virtual Element Settings
3. Robin Boundary Condition
4. Numerical Results
5. Applications

We consider,

$$u = u_i + u_r$$

with u_i the **incident wave** and u_r **the reflected wave**. We have,

$$u_i(\mathbf{x}, t) = a_i(\mathbf{x})e^{-i\sigma t} \quad \text{and} \quad u_r(\mathbf{x}, t) = a_r(\mathbf{x})e^{-i\sigma t}$$

with $\sigma = 2\pi/T_0$, the **angular frequency** and

$$a_i(\mathbf{x}) = a_{\max}e^{-i\mathbf{k}\mathbf{x}} \quad \text{with} \quad \mathbf{k} = k(\cos(\theta), \sin(\theta))$$

with θ the **incident wave angle**, $a_{\max}$ the **maximum wave amplitude**. a_r is subject to...

I) Model problem

The Helmholtz equation:

$$\begin{cases} \Delta a + k^2 a = 0, & \text{in } \Omega\,, \\ +BC. & \end{cases}$$

The Mild-Slope equation:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g\, a = 0, & \text{in } \Omega\,, \\ +BC. & \end{cases}$$

with

$$C_p = \frac{\sigma}{k} \quad \text{and} \quad C_g = \frac{1}{2}\, C_p \left[1 + kh\, \frac{1 - \tanh^2(kh)}{\tanh(kh)} \right],$$

and the wave number k, solution of the dispersion relation:

$$\sigma^2 = g\, k \tanh(kh) \quad \text{with} \quad \sigma = \frac{2\pi}{T_0},$$

where T_0 is **the wave period** and h **the depth**.

I) MODEL PROBLEM

The Helmholtz equation:

$$\begin{cases} \Delta a + k^2 a = 0, & \text{in } \Omega\,, \\ +BC. \end{cases}$$

The Mild-Slope equation:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g\, a = 0, & \text{in } \Omega\,, \\ +BC. \end{cases}$$

- Works only with flat bottoms,
- Easy-to-calculate analytical solutions.

- Works with a non-constant seabed,
- Area of validity: maximum slope of 1/3,
- Difficult to obtain an analytical solution.

1. Model problem
2. Virtual Element Settings
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- **Mesh Decomposition:** Decomposition $\{T_h\}_h$ of the domain Ω which is shape-regular. h_E the diameter of E, (x_D, y_D) the centroid of E.
- **The standard scale monomial basis:** $m_{\alpha_1,\alpha_2} = \left(\frac{x-x_D}{h_D}\right)^{\alpha_1} \cdot \left(\frac{y-y_D}{h_D}\right)^{\alpha_2}$ with $\alpha_1 + \alpha_2 \leq k$.
- **Local Projections:**
  - Local elliptic projector: $\Pi_k^{\nabla,E} : H^1(E) \to \mathbb{P}_k(E)$
  - Local L^2-projector: $\Pi_k^{0,E} : L^2(E) \to \mathbb{P}_k(E)$

- **Virtual Space:**

$$V_h^E = \Big\{ v_h \in H_0^1(\Omega) \cap C^0(\partial E) \; : \; v_h|_{\partial E} \in \mathbb{P}_k(E), \Delta v_h \in \mathbb{P}_k(E)$$
$$\left(\Pi_k^{\nabla} v_h - v_h, p\right)_{0,E} = 0, \forall p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \Big\}$$

- **Local Degrees of Freedom:**

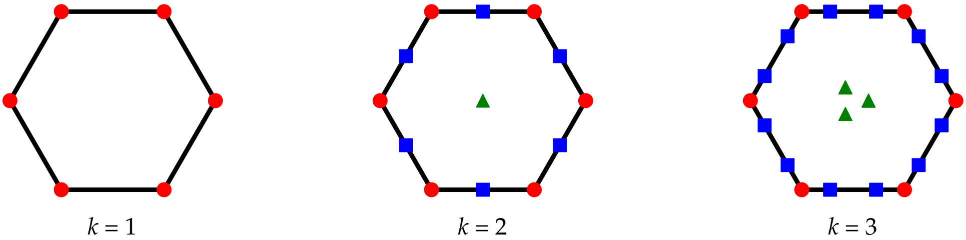


Figure 2: 2D element with ● : Summits dofs, ■ : Edges dofs, ▲ : Inner dofs.

1. Model problem

2. Virtual Element Settings

3. Robin Boundary Condition

4. Numerical Results

5. Applications

1) Variational formulation

Helmholtz:

$$\begin{cases} \Delta a + k^2 a = 0 & , \quad \text{in } \Omega\,, \\ a = -a_i & , \quad \text{on } \Gamma_D\,, \\ \dfrac{\partial a}{\partial n} + i\,k\,a = 0 & , \quad \text{on } \Gamma_{\text{Inf}}. \end{cases}$$

Mild-Slope:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g\, a = 0, & , \quad \text{in } \Omega\,, \\ a = -a_i & , \quad \text{on } \Gamma_D\,, \\ \dfrac{\partial a}{\partial n} + i\,k\,a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases}$$

$$\begin{cases} \text{find } u \in V = H_0^1(\Omega) \text{ such that} \\ \qquad a(u,v) \;=\; 0 \quad \forall v \in V, \end{cases}$$

$$\begin{aligned} a(u,v) &= \int_\Omega \Delta u\, v + k^2 \int_\Omega u\, v\,, \\ &\underset{green}{=} -\int_\Omega \nabla u \nabla v + k^2 \int_\Omega u\, v + \int_{\Gamma_{\text{Inf}}} \frac{\partial u}{\partial n} v\,, \\ &\underset{\partial u/\partial n = -iku}{=} -\int_\Omega \nabla u \nabla v + k^2 \int_\Omega u\, v - ik \int_{\Gamma_{\text{Inf}}} u\, v\,. \end{aligned}$$

III) Robin Boundary Condition

1) Variational formulation - Discrete form

$$\begin{cases} \text{find } u_h \in V_h \subset V \text{ such that} \\ \quad a_h(u_h, v_h) = 0 \quad \forall v \in V, \end{cases}$$

- $V_h \subset V$ is a finite dimensional space,
- $a_h(\cdot\,,\,\cdot) : V_h \times V_h \to \mathbb{R}$ is a discrete bilinear form approximating the continuous form $a(\cdot\,,\,\cdot)$.

$$a_h(u_h, v_h) = \sum_{E \in \Omega_h} \left[\int_E \nabla(C_p C_g \nabla u_h v_h) + \int_E k^2 C_p C_g u_h v_h \right],$$

$$\underset{\substack{1/E \int_E C_p C_g = \mathcal{A}_E \\ 1/E \int_E k^2 C_p C_g = \mathcal{B}_E}}{\approx} \sum_{E \in \Omega_h} \left[\mathcal{A}_E \int_E (\Delta u_h\, v_h) + \mathcal{B}_E \int_E u_h v_h \right],$$

$$\underset{\substack{green \\ \partial u/\partial n = -iku}}{=} \sum_{E \in \Omega_h} \left[-\mathcal{A}_E \int_E \nabla u_h \nabla v_h + \mathcal{B}_E \int_E u_h v_h - \mathbf{1}_{\Gamma_{\mathrm{Inf}} \subset E}\, i \mathcal{A}_E \int_{\Gamma_{\mathrm{Inf}}} k\, u_h v_h \right].$$

III) Robin Boundary Condition

1) Variational formulation - General case

$$\begin{cases} \Delta u + k^2\, u = 0 & , \quad \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n} + k(x,y)\, u = g(x,y) & , \quad \text{on } \Gamma_{\text{Inf}}. \end{cases}$$

$$a_h(u_h, v_h) = -\int_\Omega \nabla u_h \nabla v_h + \int_\Omega k^2\, u_h\, v_h - \underbrace{\int_{\Gamma_{\text{Inf}}} k\, u_h\, v_h}_{B}$$

$$b_h(v_h) = -\underbrace{\int_{\Gamma_{\text{Inf}}} g\, v_h}_{G}$$

By expressing in the classical shape functions basis:

$$B = \left(\int_{\Gamma_{\text{Inf}}} k(x,y)\Phi_i(x,y)\Phi_j(x,y) \right)_{i,j}$$

$$G = \left(\int_{\Gamma_{\text{Inf}}} k(x,y)\Phi_i(x,y) \right)_{i}$$

with Φ_i the classical shape functions of order k.

2) Element properties

Γ_{Inf} can be expressed as a sum of 1D elements defined by $[\xi_0, \xi_0 + \lambda]$:

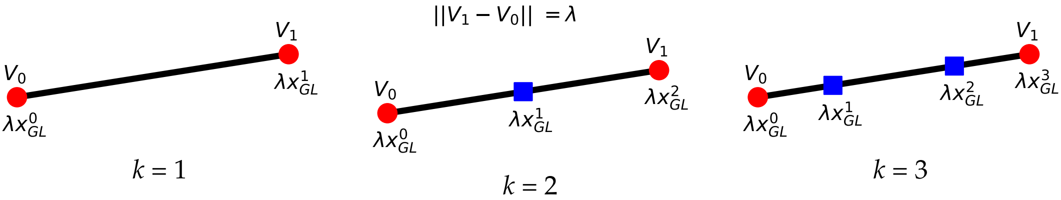


Figure 3: 1D element $[\xi_0, \xi_0 + \lambda]$ representation with
● : Summits dofs, ■ : Edges dofs.

with x_{GL}^j the $j - th$ Gauss-Lobatto quadrature point on [0,1].

3) Expression of B_{LOC} and G_{LOC}

On each $[\xi_0, \xi_0+\lambda]$ element:

$$B_{\mathrm{loc}} = \left(\int_0^{\lambda} k_*(\xi_0+\xi)\varphi_j(\xi)\varphi_i(\xi)\, d\xi \right)_{0\le i,j\le k},$$

$$G_{\mathrm{loc}} = \left(\int_0^{\lambda} g_*(\xi_0+\xi)\varphi_i(\xi)\, d\xi \right)_{0\le i\le k}.$$

- φ_i, φ_j are polynomials of order k,
- $k_*(\xi_0+\xi) = k(V_0+\xi\,\vec{t})$,
- $g_*(\xi_0+\xi) = g(V_0+\xi\,\vec{t})$,
- $\vec{t}$: tangential unit vector (V_0 to V_1).

For $i,j \in [|0,k|]$:

$$\varphi_i(\lambda\, x_{GL}^j) = \delta_j^i.$$

Using Lagrange polynomials:

$$\varphi_i(\xi) = \sum_{j=0}^{k} \delta_j^i \left(\prod_{l=0,l\neq j}^{k} \frac{\xi - \lambda x_{\mathrm{GL}}^l}{\lambda x_{\mathrm{GL}}^j - \lambda x_{\mathrm{GL}}^l} \right)$$

$$= \frac{1}{\lambda^k} \prod_{l=0,l\neq i}^{k} \frac{\xi - \lambda x_{\mathrm{GL}}^l}{x_{\mathrm{GL}}^i - x_{\mathrm{GL}}^l}$$

4) Computation of B_{LOC} and G_{LOC}

On each $[\xi_0, \xi_0 + \lambda]$ element:

$$B_{\text{loc}} = \left(\int_0^\lambda k_*(\xi_0 + \xi)\varphi_j(\xi)\varphi_i(\xi)\, d\xi \right)_{0 \le i,j \le k},$$

$$G_{\text{loc}} = \left(\int_0^\lambda g_*(\xi_0 + \xi)\varphi_i(\xi)\, d\xi \right)_{0 \le i \le k}.$$

- Approximated using Gauss-Lobatto quad of order $4k + 1$.

Case: k = constant, g = constant :

$$B = \lambda k \left(\int_0^{\mathbf{1}} \tilde{\varphi}_j(\xi)\tilde{\varphi}_i(\xi)\, d\xi \right)_{0 \le i,j \le k},$$

$$G = \lambda g \left(\int_0^{\mathbf{1}} \tilde{\varphi}_i(\xi)\, d\xi \right)_{0 \le i \le k}.$$

- $\tilde{\varphi}_i$: polynomials for a unit element $[\xi_0, \xi_0 + \mathbf{1}]$,
- Exact integration with $4k$ - 3 GL points,
- A single evaluation.

IV) Numerical Results

1) Analytical solution

We consider,

$$\begin{cases} \Delta u + k^2 u = f(x,y) & , \quad \text{in } \Omega\,, \\ u = u_{\text{exact}} & , \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\,, \\ \dfrac{\partial u}{\partial n} + i\,k\,u = g(x,y) & , \quad \text{on } \Gamma_1, \end{cases}$$

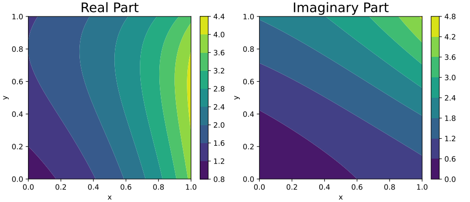
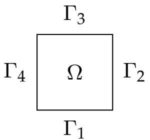


Figure 4: Real and Imaginary part of u_{exact}.

with:

$$\begin{cases} u_{\text{exact}}(x,y) = (x+y)\cdot(1+i) + \exp(x^2 + i\,y^2), \\ f(x,y) = -\Big((2x)^2 + (2\,i\,y)^2 + 2(1+i)\Big)\cdot \exp(x^2 + i\,y^2) + k^2 \cdot u_{\text{exact}}(x,y), \\ g(x,y) = (1+i) + (2\,i\,y)\cdot \exp(x^2 + i\,y^2) + i\,k \cdot u_{\text{exact}}(x,y). \end{cases}$$



IV) Numerical Results

2) Convergence of order $\mathcal{O}(h^{k+1})$

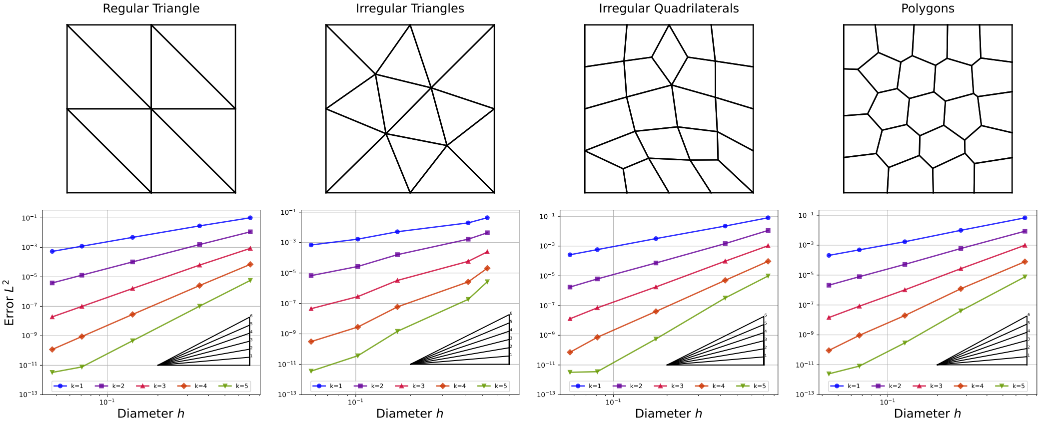


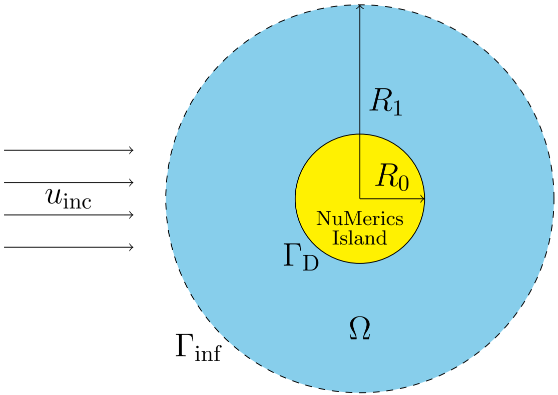
Figure 5: Convergence curves with different orders k and different types of elements

3) Interest of a Robin condition

$$\begin{cases} \Delta u + k^2 u = 0 & , \quad \text{in } \Omega\,, \\ u = -u_{\text{inc}} & , \quad \text{on } \Gamma_{\text{D}}\,, \\ \dfrac{\partial u}{\partial n} + i\,k\,u = 0 & , \quad \text{on } \Gamma_{\text{Inf}}\,. \end{cases}$$

or

$$\begin{cases} \Delta u + k^2 u = 0 & , \quad \text{in } \Omega\,, \\ u = -u_{\text{inc}} & , \quad \text{on } \Gamma_{\text{D}}\,, \\ \dfrac{\partial u}{\partial n} = 0 & , \quad \text{on } \Gamma_{\text{Inf}}\,. \end{cases}$$



IV) Numerical Results

3) Interest of a Robin condition

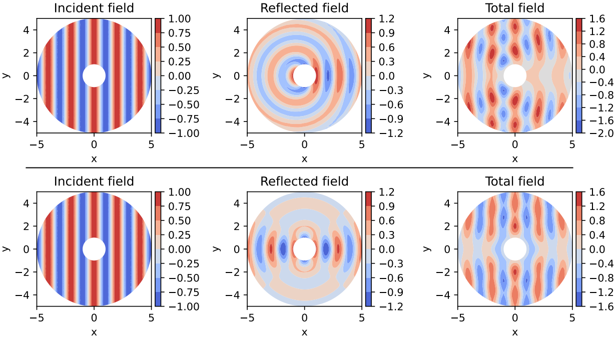
Problem conditions:

- $a_{max} = 1$ m,
- $T_0 = 20$ s,
- $\theta = 0\,^\circ$.

Points of interest:

- Disturbance of the reflected wave field.

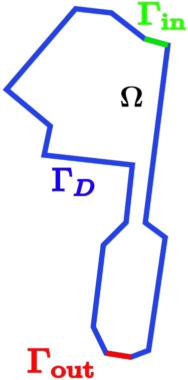
Solving the Helmholz problem with a Robin condition on Γ_{inf}



Solving the Helmholz problem with a Neuman condition on Γ_{inf}

1) Problem configuration

Port location



Port boundary

The Helmholtz or Mild-Slope equation:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g \, a = 0, & \text{in } \Omega \,, \\ a = 0, & \text{in } \Gamma_{in} \,, \\ \dfrac{\partial a}{\partial n} + ik\, a = 0, & \text{in } \Gamma_{out} \,, \\ a = \gamma \, a_i & \text{in } \Gamma_D \,. \end{cases}$$

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V) Applications

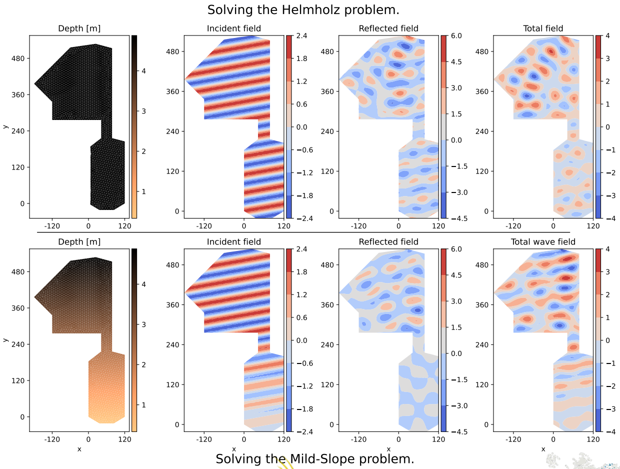
2) Slope sensitivity, Helmholtz vs Mild-Slope

Problem conditions:

- $a_{max} = 2$ m,
- $T_0 = 8$ s,
- $\theta = 280$ °.

Points of interest:

- Eigenmode position.



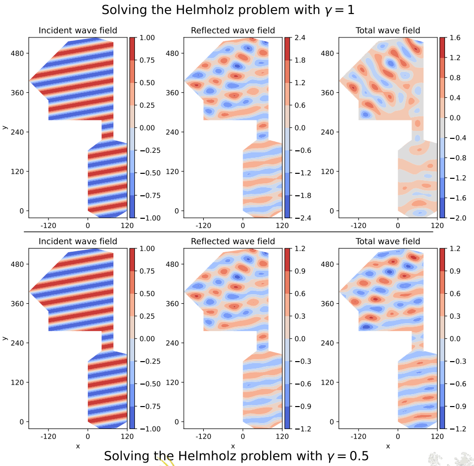
3) Reflection coefficient sensitivity γ

Problem conditions:

- $a_{max} = 1$ m,
- $T_0 = 8$ s,
- $\theta = 280$ °.

Points of interest:

- Eigenmode position,
- Amplitude of reflected wave field.



4) Results at different orders k

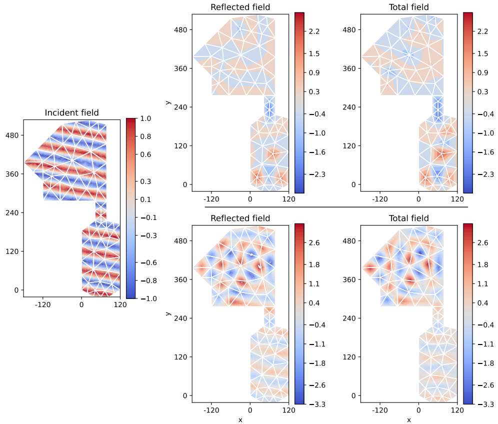
Problem conditions:

- $a_{max} = 1$ m,
- $T_0 = 8$ s,
- $\theta = 250\,°$.

Points of interest:

- Eigenmode position.

Solving the Helmholz problem with k=1



Solving the Helmholtz problem with k=5

Grazie per l'attenzione!